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SUPERCONFORMAL ASPECTS OF $d = 11$ SUPERGRAVITY

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We discuss superconformal aspects of supergravity in eleven dimensions. We suggest that the Poincaré theory is obtained by a compensating mechanism using a scalar superfield.

Superconformal concepts help to clarify the off-shell structure of Poincaré supergravity [‡1]. In this letter, we consider the application of this idea to supergravity in eleven dimensions ($d = 11$). We construct part of an off-shell multiplet which contains the superconformal gauge fields (the linearized Weyl multiplet). Using a scalar multiplet as a compensating multiplet, we obtain part of the off-shell Poincaré theory. These results are in agreement with those of ref. [2].

The Weyl multiplet is defined to be the smallest off-shell multiplet containing supergravitational spin 2 and spin 3/2 representations. It should contain the states of the on-shell Poincaré theory [3] as a massless submultiplet, to which it can be restricted by imposing appropriate conditions. The usual procedure to construct the Weyl multiplet is to consider a suitable matter multiplet and to apply supercurrent techniques [4]. Obviously, this method fails in $d = 11$ due to the absence of matter multiplets. We therefore start with irreducible massive spin 2 and spin 3/2 representations, and attempt to close the superconformal commutator algebra on these fields. This reveals the need for spin 1 fields in the form of antisymmetric tensors. In addition, we are forced to introduce a spinor λ which satisfies a differential constraint, as was also the case in the $d = 10$ Weyl multiplet [5]. Off-shell supermultiplets in $d = 11$ are large, the smallest contains at least 2^{16} field components. Therefore, we do not expect to be able to complete this ad hoc construction of the Weyl multiplet. We do find, however, that the low spin part that we obtain, is essentially insensitive to the higher spin fields which are undoubtedly required to close the algebra. Our partial results will therefore remain valid if eventually the full multiplet is obtained by other means.

Let us now consider the construction of the Weyl multiplet in more detail. We start with elfbein and gravitino fields e_μ^a and ψ_μ. To separate the irreducible spin 2 and spin 3/2 states we introduce local dilatations D, S-supersymmetry transformations and conformal boosts K, and their respective gauge fields b_μ, ϕ_μ and f_μ^a. As a starting point, we take for e_μ^a, ψ_μ and b_μ the standard transformation rules:

$$\delta e_\mu^a = \tfrac{1}{2}\bar{\epsilon}\Gamma^a\psi_\mu - \Lambda_D e_\mu^a,$$

$$\delta\psi_\mu = \mathcal{D}_\mu(\omega, b)\epsilon - \tfrac{1}{2}\Lambda_D\psi_\mu - \Gamma_\mu\eta,$$

$$\delta b_\mu = \tfrac{1}{2}\bar{\epsilon}\phi_\mu + \partial_\mu\Lambda_D - \tfrac{1}{2}\bar{\eta}\psi_\mu + e_\mu^a\Lambda_{Ka}. \tag{1}$$

The derivatives $\mathcal{D}_\mu$ are covariant with respect to con-

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[‡1] For a review of superconformal methods, see ref. [1].

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formal transformations. The covariant curvatures are

$$R^a_{\mu\nu}(P) = \mathcal{D}_{[\mu} e_{\nu]}{}^a, \quad R_{\mu\nu}(Q) = \mathcal{D}_{[\mu}\psi_{\nu]} - \Gamma_{[\mu}\phi_{\nu]},$$

$$R^{ab}_{\mu\nu}(M) = \partial_{[\mu}\omega_{\nu]}{}^{ab} - \omega_{[\mu}{}^{ac}\omega_{\nu]}{}^{cb} - 2f_{[\mu}{}^{[a}e_{\nu]}{}^{b]}, \tag{2}$$

and satisfy the usual conventional constraints

$$R^a_{\mu\nu}(P) = 0, \quad \Gamma^\mu R_{\mu\nu}(Q) = 0, \quad R^{ab}_{\mu\nu}(M)e^\nu_b = 0. \tag{3}$$

Because of (3), the gauge fields ω^{ab}_μ, ϕ_μ and f^a_μ are dependent. The commutator of two Q transformations must yield a supercovariant translation and is allowed to give other field dependent transformations. With the transformations (1), this is the case on the elfbein, but not on ψ_μ. If we keep for the moment the transformation of e^a_μ as it is, the only allowed modification in $\delta\psi_\mu$ (modulo field dependent S transformations) is by terms containing three- and four-index antisymmetric tensors. These generate field dependent Lorentz transformations in the algebra on e^a_μ. With the transformations

$$\delta\psi^{\text{extra}}_\mu = (\Gamma_\mu\Gamma^{abcd} - 3\Gamma^{abcd}\Gamma_\mu)\epsilon N_{abcd},$$

$$\delta N_{abcd} = \tfrac{1}{96}\bar{\epsilon}\Gamma_{[ab}R_{cd]}(Q), \tag{4}$$

we correctly generate the supercovariant translation on ψ_μ. The coefficients in (4) are chosen such that the algebra does not contain a field dependent S transformation. A three-index tensor N_{abc} is not required at this stage. This result could have been anticipated from the known (on-shell) closure of the algebra of super-Poincaré transformations [3].

Next we gather the ϕ_μ terms in the commutator of two Q transformations on ψ_μ. These arise from the variation of ω^{ab}_μ and b_μ, and from part of $R_{\mu\nu}(Q)$. In order to cancel these terms, we are forced to introduce a spinor λ which transforms inhomogeneously under S transformations, i.e. $\delta_S\lambda = \eta$. We then allow transformations of N_{abc} and N_{abcd} into

$$D_\mu\lambda = \mathcal{D}_\mu\lambda - \phi_\mu. \tag{5}$$

These transformations can be chosen in such a way that in the commutator on ψ_μ one finds only λ-dependent Q transformations and $\not{D}\lambda$-dependent S transformations. The λ-dependent Q transformations give rise to new terms containing ϕ_μ, which then precisely cancel against the original ϕ_μ terms. Note that explicit ϕ_μ terms in the transformation of $N_{(3)}$ and $N_{(4)}$ are not allowed. Such terms would give rise to gauge transformations of $N_{(3)}$ and $N_{(4)}$, as can be seen from the $[Q,S]$ commutator. However, such a gauge transformation applied to $\delta\psi_\mu$ does not yield a supersymmetry transformation.

One should then show that the algebra closes on λ as well. It turns out, however, that it is impossible to obtain the $\not{D}\lambda$-dependent S transformation on λ itself, and we must therefore conclude that λ is not an independent field, but is determined in terms of ψ_μ by the differential constraint

$$\not{D}\lambda = 0. \tag{6}$$

The constraint (6) is analogous to a similar condition in the $d = 10$ Weyl multiplet [5]. The difference with $d = 10$ is that in $d = 11$ the constraint (6) is not required to obtain closure on ψ_μ, but only for closure on λ itself. With the constraint (6) the linearized transformation rules of λ are determined by those of ψ_μ. We can write the linearized form of (6) as:

$$\lambda = -\tfrac{1}{10}(1/\Box)\not{\partial}\Gamma^{\mu\nu}\partial_\mu\psi_\nu, \tag{7}$$

which shows that (as in $d = 10$) any attempt to formulate the Weyl multiplet without the dependent field λ necessarily involves nonlocal terms. It is posible to relax the constraint (6) by adding another multiplet. One introduces tensors $T_{(3)}$ and $T_{(4)}$ in $\delta\psi_\mu$, whose variation into $\not{D}\lambda$ cancels the field dependent S transformation. This will result in a larger, reducible field representation without differential constraints.

The linearized transformation rules of the independent fields are at this stage

$$\delta e^a_\mu = \tfrac{1}{2}\bar{\epsilon}\Gamma^a\psi_\mu,$$

$$\delta\psi_\mu = \mathcal{D}_\mu\epsilon + (\Gamma_\mu\Gamma^{abcd} - 3\Gamma^{abcd}\Gamma_\mu)\epsilon N_{abcd} + (5\Gamma_\mu\Gamma^{abc} + 9\Gamma^{abc}\Gamma_\mu)\epsilon N_{abc} - \Gamma_\mu\eta,$$

$$\delta N_{abc} = \tfrac{1}{64}\bar{\epsilon}\Gamma_{[ab}D_{c]}\lambda,$$

$$\delta N_{abcd} = \tfrac{1}{96}\bar{\epsilon}\Gamma_{[ab}R_{cd]}(Q) + \tfrac{1}{192}\bar{\epsilon}\Gamma_{[abc}D_{d]}\lambda. \tag{8}$$

The algebra contains a λ-dependent Q transformation, but no field dependent S transformations. With (7) and (8) we can now calculate the transformation of λ. If any nonlocal terms remain in $\delta\lambda$ this signals the need to introduce other dependent fields. We find that indeed a dependent scalar σ, as well as dependent tensors C_{abc} and C_{abcd} are required. They are given by the relations

$$\Box\sigma = -\tfrac{1}{10} R(e),$$

$$\Box C_{abc} = -\tfrac{1}{5}(7\Box N_{abc} - 27\,\partial_{[a}\partial^{d} N_{bc]d}),$$

$$\Box C_{abcd} = -\tfrac{1}{5}(2\Box N_{abcd} + 12\,\partial_{[a}\partial^{f} N_{bcd]f}). \tag{9}$$

With (7), (8) and (9) the transformation rules of the dependent fields can be determined. We find:

$$\delta\sigma = \tfrac{1}{2}\bar{\epsilon}\lambda + \Lambda_{\mathrm{D}},$$

$$\delta\lambda = \tfrac{1}{2}\not{\mathrm{D}}\sigma\epsilon + \Gamma^{abc}\epsilon C_{abc} + \Gamma^{abcd}\epsilon C_{abcd} + \eta,$$

$$\delta C_{abc} = -\tfrac{1}{64}\bar{\epsilon}\Gamma_{[ab}\mathrm{D}_{c]}\lambda - \tfrac{1}{160}\bar{\epsilon}\Gamma_{[a}R_{bc]}(Q),$$

$$\delta C_{abcd} = -\tfrac{1}{192}\bar{\epsilon}\Gamma_{[abc}\mathrm{D}_{d]}\lambda$$

$$- (6\cdot 160)^{-1}\bar{\epsilon}\Gamma_{[ab}R_{cd]}(Q). \tag{10}$$

The algebra now closes on e_μ^a and ψ_μ, but not on $N_{(3)}$ and $N_{(4)}$. The transformation rules of these fields therefore require additional terms, containing spinors with antisymmetrized Lorentz indices. These terms will then, via (9), determine whether or not more dependent fields are required. A straightforward calculation shows that any new terms in the transformations of $N_{(3)}$ and $N_{(4)}$ do not cancel in the commutator calculated on ψ_μ. The only way to obtain such a cancellation is to allow the appearance of new spin 2 fields in the transformation of ψ_μ itself.

With additional spin 2 fields it is not difficult to envisage how closure of the algebra on $N_{(3)}$, $N_{(4)}$ and ψ_μ might be achieved. We will not do this extremely complicated calculation, but we will indicate that its result does not modify the reasoning which led to (8), (9) and (10). Depending on the dimension of the spin 2 field, the variation of ψ_μ may involve a derivative. If it does, our calculation leading to (4) changes if the variation of the new spin 2 field contains ψ_μ. However, such a variation implies that this spin 2 field is not auxiliary in the corresponding Poincaré theory. If there is no derivative in the variation of ψ_μ, the spin 2 field might go to ϕ_μ, since ϕ_μ is proportional to the Poincaré field equation of ψ_μ. However, the argument which excludes a ϕ_μ variation of $N_{(3)}$ and $N_{(4)}$ also applies here. Therefore, the inclusion of spin 2 fields besides the elfbein, which is certainly required to extend (8) to the full Weyl multiplet, does not modify the terms already present in (8).

At this point it is not clear, which and how many high spin fields will be required, but the anticommuting character of the supersymmetry generators ensures that the iterative procedure will terminate. Our construction will thus generate a finite supermultiplet, and its lowest dimensional field may be identified with the first component of the corresponding superfield.

Of course, the inclusion of higher spin fields will modify (9) with terms containing these fields. As a consequence, $C_{(3)}$ and $C_{(4)}$ will acquire inhomogeneous gauge transformations, analogous to the D and S transformations of σ and λ. These play an important role in the restriction of the Weyl multiplet to the on-shell Poincaré multiplet.

Let us briefly indicate how this on-shell submultiplet is obtained. First we set the dependent fields σ, λ, $C_{(3)}$ and $C_{(4)}$ equal to zero by gauge choices. We then impose the gravitino field equation, and investigate its consequences. We find that the Ricci-tensor must vanish and that

$$N_{abc} = 0, \quad \partial^a N_{abcd} = 0, \quad \partial_{[a}N_{bcde]} = 0, \tag{11}$$

so that $N_{(4)}$ describes the spin 1 degrees of freedom of the on-shell Poincaré multiplet. Note that it is crucial that $C_{(3)}$ and $C_{(4)}$ develop inhomogeneous gauge transformations due to new spin 2 fields. The transformation of $C_{(3)}$ and $C_{(4)}$ to $R(Q)$ then becomes a field dependent gauge transformation, which is irrelevant to the on-shell Poincaré submultiplet.

Let us now apply these results to obtain information about the off-shell structure of the $d = 11$ Poincaré supergravity theory. This requires a compensating multiplet to break some of the superconformal symmetries. The most natural candidate for such a compensating multiplet is the $d = 11$ scalar multiplet Φ. The $d = 11$ scalar superfield $\Phi(x,\theta)$ can be expanded as follows:

$$\Phi(x,\theta) = A + \bar{\theta}\xi + \bar{\theta}\Gamma^{(4)}\theta B_{(4)} + \bar{\theta}\Gamma^{(3)}\theta B_{(3)} + \bar{\theta}\theta B + \ldots, \tag{12}$$

where θ is a Majorana spinor. It contains 2^{32} components. The fermionic representations all form multiples of 32, since the basic spinor representation of

Table 1
The first sectors of the scalar superfield in $d = 11$.

$n\theta$		SO(11) representation	
0	A	[0,0,0,0,0]	1
1	λ	$[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	1×32
2	B	[0,0,0,0,0]	1
	$B_{(3)}$	[1,1,1,0,0]	165
	$B_{(4)}$	[1,1,1,1,0]	330
3	χ	$[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	1×32
	$\chi_{(2)}$	$[\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$	44×32
	$\chi_{(3)}$	$[\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2}]$	110×32
4	C	[2,2,1,1,1]	17160
		[2,2,2,0,0]	7865
		[2,2,1,0,0]	5005
		[2,1,1,1,1]	4290
		[2,2,0,0,0]	1144
		[1,1,1,1,0]	330
		[1,1,1,0,0]	165
		[0,0,0,0,0]	1

SO(11) has dimension 32. In table 1 we present the SO(11) representations contained in Φ, up to and including the 4-θ sector. One can find these representations, and the ones contained in the higher sectors, by using the expansion of the $d = 10$ scalar superfield [6] [‡2].

We find that the $d = 11$ scalar superfield is reducible. In the 2-θ sector one can impose two inequivalent constraints:

$$\bar{\mathrm{D}}\mathrm{D}\Phi(x,\theta) = 0, \tag{13}$$

$$\partial^c \bar{\mathrm{D}}\Gamma_{abc}\mathrm{D}\Phi(x,\theta) = \partial_{[a}\bar{\mathrm{D}}\Gamma_{bcde]}\mathrm{D}\Phi(x,\theta) = 0. \tag{14}$$

The first constraint leads to a multiplet with highest spin 8, while the second constraint projects out a multiplet of 2×2^{16} components, with highest spin equal to 4.

The transformation rules of the fields in the first few sectors of Φ are given by

$$\delta A = \bar{\epsilon}\xi,$$

$$\delta\xi = \tfrac{1}{4}\not{\partial}A\epsilon + (B + \Gamma^{(3)}B_{(3)} + \Gamma^{(4)}B_{(4)})\epsilon,$$

$$\delta B = +\tfrac{1}{64}\,\bar{\epsilon}\not{\partial}\xi + \bar{\epsilon}\chi,$$

$$\delta B_{abc} = -(2\cdot 192)^{-1}\epsilon\not{\partial}\Gamma_{abc}\xi + \tfrac{1}{90}\bar{\epsilon}\,\Gamma_{abc}\chi + \bar{\epsilon}(\chi_{abc} + \Gamma_{[a}\chi_{bc]}),$$

$$\delta B_{abcd} = (8\cdot 192)^{-1}\,\bar{\epsilon}\not{\partial}\Gamma_{abcd}\,\xi - (4\cdot 90)^{-1}\,\bar{\epsilon}\Gamma_{abcd}\chi - \bar{\epsilon}\Gamma_{[a}(\chi_{bcd]} - \tfrac{1}{6}\Gamma_{[b}\chi_{cd]}),$$

$$\delta\chi = \tfrac{15}{64}\not{\partial}B\epsilon - \tfrac{3}{64}\Gamma^{ab}\epsilon\partial^\lambda B_{\lambda ab} - \tfrac{1}{128}\Gamma^{a_1-a_5}\epsilon\partial_{[a_1}B_{a_2-a_5]} + \ldots,$$

$$\begin{aligned}\delta\chi_{ab} = {} & \tfrac{3}{20}\,\epsilon\partial^\lambda B_{\lambda ab} + (7\cdot 32)^{-1}\Gamma^{\lambda\rho}\epsilon(9\partial_{[a}B_{b]\lambda\rho} - 33\partial_\lambda B_{\rho ab}) \\ & - \tfrac{3}{28}\Gamma^\lambda\epsilon\partial^\rho B_{\lambda\rho ab} + \tfrac{3}{140}\Gamma^{\lambda\rho\sigma}\epsilon(\partial_{[a}B_{b]\lambda\rho\sigma} - \tfrac{7}{2}\partial_\lambda B_{\rho\sigma ab}) - \text{traces} + \ldots,\end{aligned}$$

$$\begin{aligned}\delta\chi_{abc} = {} & 39(7\cdot 64)^{-1}\Gamma^\lambda\epsilon\partial_\lambda B_{abc} + 9(7\cdot 64)^{-1}\Gamma^\lambda\epsilon\partial_{[a}B_{bc]\lambda} - \tfrac{13}{56}\epsilon\partial^\lambda B_{\lambda abc} \\ & + \tfrac{5}{24}\Gamma^{\lambda\rho}\epsilon(\partial_\lambda B_{\rho abc} - \tfrac{3}{7}\partial_{[a}B_{bc]\lambda\rho}) - \text{traces} + \ldots .\end{aligned} \tag{15}$$

Here the dots indicate contributions from the next sector, which we have not calculated. The algebra closes on the fields A, ξ and the B's.

In order to describe the lowest order coupling to conformal supergravity we need only replace ordinary derivatives by supercovariant ones. We assign a Weyl $w = 1$ to the scalar component A. In lowest order the spinor ξ then transforms under S supersymmetry as

$$\delta_S\xi = +\tfrac{1}{2}A\eta, \tag{16}$$

and hence

$$\not{D}\xi = \not{\partial}\xi - \tfrac{1}{2}A\Gamma\cdot\phi. \tag{17}$$

In terms of the Poincaré field equation $R^\mu = \Gamma^{\mu\nu\rho}\partial_\nu\psi_\rho$ the S-gauge field ϕ_μ is given by

$$\phi_\mu = \tfrac{1}{9}(R_\mu - \tfrac{1}{10}\Gamma_\mu\,\Gamma\cdot R). \tag{18}$$

[‡2] Such an analysis is suggested in ref. [7].

We are now ready to apply the compensating mechanism. To break the invariance under dilatations we adjust the scalar A to a constant, while the invariance under S supersymmetry and K transformations is broken by setting ξ and b_μ equal to zero. This leads to the following decomposition rule for the Poincaré supersymmetry transformations:

$$\delta^{\text{Poincaré}}(\epsilon) = \delta_{\mathrm{Q}}(\epsilon) + \delta_{\mathrm{S}}(\eta = -2(B + \Gamma^{(3)}B_{(3)} + \Gamma^{(4)}B_{(4)})\epsilon) + \delta_{\mathrm{K}}(\Lambda_{\mathrm{K}\mu} = -\tfrac{1}{2}\bar{\epsilon}\phi_\mu). \tag{19}$$

After these gauge conditions have been imposed the linearized Q transformations of the B's are given by

$$\delta B = \bar{\epsilon}\chi + (80\cdot 144)^{-1}\,\bar{\epsilon}\Gamma\cdot R,$$

$$\delta B_{abc} = \tfrac{1}{90}\,\bar{\epsilon}\Gamma_{abc}\chi + \bar{\epsilon}\Gamma_{[a}\chi_{bc]} + \bar{\epsilon}\chi_{abc} + (48\cdot 144)^{-1}(6\bar{\epsilon}\Gamma_{[ab}R_{c]} - \tfrac{1}{2}\,\bar{\epsilon}\Gamma_{abc}\,\Gamma\cdot R),$$

$$\delta B_{abcd} = -\tfrac{1}{360}\,\bar{\epsilon}\Gamma_{abcd}\chi + \tfrac{1}{6}\,\bar{\epsilon}\Gamma_{[ab}\chi_{cd]} - \bar{\epsilon}\Gamma_{[a}\chi_{bcd]} + (192\cdot 144)^{-1}(8\bar{\epsilon}\Gamma_{[abc}R_{d]} - \tfrac{7}{10}\,\bar{\epsilon}\Gamma_{abcd}\,\Gamma\cdot R). \tag{20}$$

The Poincaré transformations follow from (8) and the decomposition rule (19), e.g.

$$\delta\psi_\mu = \mathcal{D}_\mu(\omega)\epsilon + (\Gamma_\mu\Gamma^{abcd} - 3\Gamma^{abcd}\Gamma_\mu)\epsilon N_{abcd} + (5\Gamma_\mu\Gamma^{abc} + 9\Gamma^{abc}\Gamma_\mu)\epsilon N_{abc} + 2\Gamma_\mu(B + \Gamma^{(3)}B_{(3)} + \Gamma^{(4)}B_{(4)})\epsilon. \tag{21}$$

These results are in agreement with those obtained by Van Proeyen [2]. We note that our $N_{(4)}$ is a combination of his $N_{(4)}$ and $F_{(4)}$, the other independent combination being absent in our field representation. However, we have fields that satisfy differential constraints. As indicated before we can eliminate these constraints by adding an additional multiplet.

We have shown that superconformal methods can be used to elucidate the structure of $d = 11$ supergravity. In $d = 11$ there are no matter multiplets, so that the systematic approach through the construction of a supercurrent is not applicable. Nevertheless, the requirement that the superconformal algebra closes modulo gauge transformations leads to a number of unambiguous results. We find that the Weyl multiplet involves differential constraints on would-be compensating fields, a concept first encountered in $d = 10$ conformal supergravity. It is obviously important to unravel further the multiplet structure of conformal supergravity, and to identify the Weyl multiplet with a (constrained) superfield.

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